EMERTON'S APPROACH TO FONTAINE-MAZUR FOR $GL_2(\mathbb{Q})$. THE PROMODULARITY THEOREM.

TOM LOVERING

Abstract.

Contents

1. Introduction

Throughout this note we take $p > 2$, and let χ denote the cyclotomic character, ϵ the Teichmuller lift of its reduction mod p. We fix a coefficient ring $\mathcal{O}$ with fraction field E which is simultaneously finite over $\mathbb{Z}_p$ and large enough to contain any numbers we might care about.

In this talk we will assemble the final ingredient needed in Emerton's proof of the following conjecture of Fontaine and Mazur.

Conjecture 1.1 (Fontaine-Mazur conjecture for $GL_2(\mathbb{Q})$)**.** *If ρ is an odd*[1] *irreducible two dimensional $G_{\mathbb{Q}}$-representation unramified outside a finite set of primes and de Rham at p with distinct Hodge-Tate weights, then ρ is a twist of a Galois representation coming from a classical modular form of weight $k \geq 2$.*

[1] The "odd" hypothesis is conjectured (known?) to be redundant given the other hypotheses.

Given the success of Taylor-Wiles, the most natural approach to this conjecture is the following. By Serre's conjecture, we know that the residual representation $\bar{\rho}$ is modular. We might hope to cook up some local conditions cutting out a Galois deformation ring that includes our modular point and also the point ρ, and try to prove an $R = \mathbb{T}$ theorem. For example, if ρ is crystalline and of small weight, this is rather standard and done in Gee's AWS notes [6]. However, for more complex p-adic Hodge theory conditions this approach leads to a difficult problem of controlling the singularities of the deformation rings, the essence of which is captured in a statement known as the Breuil-Mezard conjecture. Nevertheless, using p-adic Langlands to relate these deformation rings to deformation rings of representations of $GL_2(\mathbb{Q}_p)$, Kisin was able to successfully use this strategy to resolve the Breuil-Mezard and Fontaine-Mazur conjectures in many cases.

Emerton's approach is fundamentally different from this and completely avoids directly studying these complicated deformation rings. The theorem he proves is the following.

Theorem 1.2 (Fontaine-Mazur, with technical restrictions)**.** *Let ρ be a continuous irreducible odd two dimensional $G_{\mathbb{Q}}$-representation over E, unramified outside a finite set of primes and de Rham at p with distinct Hodge-Tate weights. Suppose its reduction mod p satisfies the conditions*

1. *$\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible,*
2. *$\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is not isomorphic to a twist of*

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & * \\ 0 & \bar{\epsilon} \end{pmatrix}.$$

Then ρ arises as a twist of a representation attached to a cuspidal Hecke eigenform of weight $k \geq 2$.

Recall that we have spent lots of time proving our main theorem that for such ρ if we additionally can show it is *promodular*, then there is an embedding of its image under p-adic Langlands $B(\rho)$ into $Hom_{E[G_{\mathbb{Q}}]}(\rho, \hat{H}^1_E)$. By a crucial property of p-adic Langlands, that ρ is de Rham tells us $B(\rho)_{l-alg} \neq 0$, whence $Hom_{E[G_{\mathbb{Q}}]}(\rho, \hat{H}^1_E)_{l-alg} \neq 0$.

But in his previous papers [5],[4, 7.4.2], Emerton identified the locally algebraic vectors in $\hat{H}^1_E$ as follows.

Theorem 1.3. *There is a natural $G_{\mathbb{Q}} \times GL_2(\mathbb{A}^\infty)$-equivariant isomorphism*

$$\bigoplus_{k\geq 2, n\in\mathbb{Z}} H^1(\mathcal{V}_k) \otimes (Sym^{k-2}(E^2))^\vee \otimes_E E(n) \xrightarrow{\cong} \hat{H}^1_{E,l-alg},$$

(where $GL_2(\mathbb{A}^\infty)$ acts on the symmetric power by projection to $GL_2(\mathbb{Q}_p)$ in the obvious fashion, and on $E(n)$ by projection to $GL_2(\mathbb{Q}_p)$ and the character $\det^n$).

In particular, we see that the only $G_{\mathbb{Q}}$-isotypical components with locally algebraic vectors are those coming from twists of representations associated to classical modular forms. Putting everything together, we conclude that if ρ is promodular as above (in particular de Rham) then it is a twist of a representation coming from a classical modular form.

Thus, to conclude the Fontaine-Mazur conjecture, the theorem we are reduced to proving is the following [3, 1.2.3].

Theorem 1.4 ("Deformed Serre conjecture"). *Let ρ be a continuous irreducible odd two dimensional $G_{\mathbb{Q}}$-representation over E, unramified outside a finite set of primes. Suppose its reduction mod p satisfies the conditions*

(1) $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ *is absolutely irreducible,*
(2) $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ *is not isomorphic to a twist of*

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ \textit{or} \ \begin{pmatrix} 1 & * \\ 0 & \bar{\epsilon} \end{pmatrix}.$$

Then V is promodular with tame level divisible only by primes at which ρ is ramified.

The strategy we follow is to form the 'big' deformation ring R_Σ of deformations of $\bar{\rho}$ that are unramified outside Σ, but with no p-adic Hodge theory condition imposed at all. In contrast to usual Taylor-Wiles situations, this gives us a positive-dimensional space. Our task can then be reduced to showing the modular points are Zariski dense[2]in this space.

By Serre's conjecture, it contains a modular point that can be arranged to have pleasant weight and level and so for whose p-adic Hodge type we can prove a small $R = \mathbb{T}$ theorem without too much difficulty. We then leverage the good behaviour of R_Σ at this modular point to deduce good geometric properties of $\operatorname{Spec} R_\Sigma$ and in particular find a smooth open 3-dimensional ball containing the modular point. Then working inside this ball (after maybe switching modular points to make sure everything is suitably set up) we can cleanly use Coleman's beautiful theory of p-adic analytic families of overconvergent modular forms to establish Zariski-density.

In these notes we extract the main ideas needed to make these arguments work, but make no effort to do the relatively easy but tedious 'admin' required to arrange for the necessary hypotheses at each stage to be satisfied, referring the interested reader to where Emerton does this [3, 7.3] and Bockle's arguments, in particular [1, 5.7].

2. Geometry of big deformation rings

In this section we present Bockle's description [1, 4] of most big global deformation rings of two dimensional $G_{\mathbb{Q}}$-representations. Crucially, we will show they are equidimensional, reduced, and have a dense open formally smooth locus containing a good supply of modular points. We will need as inputs $R = \mathbb{T}$ theorems of roughly the strength proved in Gee's AWS notes [6], and some cohomological "local-global" arguments of Bockle, but otherwise we are really just doing delicate commutative algebra.

[2]Once we have Zariski density, and at an early stage in the proof we will also show that R_Σ is reduced, then the map to the corresponding 'big Hecke algebra' $R_\Sigma \twoheadrightarrow \mathbb{T}_\Sigma$ has nilpotent, hence zero kernel.

2.1. **Statement of results.** Let $\bar{\rho}$ be a mod π Galois representation, assumed odd. We also add in the standard technical condition that $\bar{\rho}|_{G(\zeta_p)}$ is absolutely irreducible. Finally, to simplify the exposition we will assume $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is also absolutely irreducible, but the same methods can be used (and are in [1]) for the reducible case under mild restrictions (matching those of Emerton). We let Σ be a finite set of primes containing the primes $\Sigma_0 = Ram(\bar{\rho}) \cup \{p\}$, and let R_Σ be a Galois deformation ring parameterising lifts which are unramified away from Σ, but with no local condition imposed at p.[3]

Theorem 2.1. *Suppose one can impose a local condition $\mathcal{L}$ at p which cuts out a modular point and such that the associated deformation ring R^s_Σ is finite flat over $\mathcal{O}$, and etale on the generic fibre. Moreover, suppose the local constrained deformation ring R^s_l is isomorphic to $\mathcal{O}[[T]]$. Then*

(1) *There is a finite flat map $\beta : \mathcal{O}[[X_1, X_2, X_3]] \to R_\Sigma$ such that $(\beta(X_1), \beta(X_2), \beta(X_3))$ is the kernel of $R_\Sigma \twoheadrightarrow R^s_\Sigma$.*
(2) *The space $\operatorname{Spec} R_\Sigma$ is equidimensional without embedded components.*
(3) *Let P_i be the closed points of $R^s_\Sigma[1/p]$. Then there is a dense open subset $V \subset \operatorname{Spec} R_\Sigma$ containing all the P_i such that V is formally smooth over $\mathcal{O}$. Moreover, each component contains some P_j, and is reduced.*

The hypotheses of the theorem can be verified in all cases we will consider by using Serre's conjecture (including a weight lowering statement) to produce a modular point whose representation is Fontaine-Laffaille at p. Imposing this Fontaine-Laffaille condition at p gives a formally smooth relative dimension 1 deformation problem (a standard result generalising Ramakrishna's theorem on flat deformation rings), and it is possible to prove a classical $R = \mathbb{T}$ theorem for R^s_Σ (as in [6]), whence the hypotheses on R^s_Σ follow from well-known features of the corresponding Hecke algebras.

2.2. **Finite flat presentation over a 3-ball.** Firstly, we fix a determinant η (and with everything set up appropriately, every deformation in R^s_Σ has this as its determinant. This gives rise to a ring $R_\Sigma \twoheadrightarrow R^\eta_\Sigma$ such that in fact $R_\Sigma \overset{\cong}{\to} R^\eta_\Sigma \hat{\otimes} \mathcal{O}[[d]]$, which is a bit more technically amenable and still contains R^s_Σ.

First note that $R^\eta_\Sigma \to R^s_\Sigma$ has kernel generated by two elements r_1, r_2. Note that since it was obtained by imposing a local condition at p, it is a pushout of a map of universal local deformation rings $R_p \to R^s_p$. By hypothesis, $R^s_p \overset{\cong}{\to} \mathcal{O}[[T]]$ and a local Galois cohomology calculation (H^1 is 3-dimensional and H^2 vanishes) tells us $R_p \overset{\cong}{\to} \mathcal{O}[[T_1, T_2, T_3]]$. Since maps between regular Noetherian local rings are complete intersections, this has kernel generated by two elements, whence the claim about the global rings.

The key step now is that one can also exhibit a presentation $R = \mathcal{O}[[x_1, ..., x_n]] \twoheadrightarrow R^\eta_\Sigma$ whose kernel is generated by $n - 2$ elements $g_1, ..., g_{n-2}$. To do this one uses Bockle's "local-global principle" [2] which uses judiciously chosen primes (a la Taylor-Wiles) to

[3] There are some additional technicalities involving fixing orders of local determinants, contributing an easily controlled relative dimension zero piece to the deformation ring, which one should worry about at this point but which we will sweep under the carpet in the interests of presentation.

present global deformation rings using nice local deformation rings, and concludes that the minimum number of relations one needs in such a presentation is the same as one needs to present all the local rings in question. But with no local conditions really present, the rings involved are all well-behaved and standard computations in Galois cohomology give that only $n-2$ generators are needed.

By a dimension argument, and since R^s_Σ/p is finite, we conclude immediately that $g_1, ..., g_{n-2}, r_1, r_2, l$ is a regular sequence[4]. In particular, R^η_Σ is a complete intersection ring and one has a regular sequence r_1, r_2, l in it.

We now use the following commutative algebra fact, proved using the Koszul complex.

Lemma 2.2. *Let R be a complete noetherian local $\mathcal{O}$-algebra of dimension $m+1$. Then $f_1, ..., f_m, l$ are a regular sequence if and only if the map they induce*

$$\mathcal{O}[[x_1, ...x_m]] \to R$$

is finite flat.

We give a proof of this fact in the appendix. This in hand, we have proved part (1) of the theorem. Part (2) of the theorem is now an immediate consequence by standard commutative algebra (any local intersection ring has these properties).

2.3. **The unramified locus.** With this in hand, let $A = \mathcal{O}[[T_1, T_2, T_3]]$, and we make the following definition. Let $M = \Omega^1_{R_\Sigma/A}$ viewed as an A-module. Since $\beta : A \to R_\Sigma$ is finite, M is a finite A-module, and its support is a closed set of $\operatorname{Spec} A$. Moreover, its support is not the whole of $\operatorname{Spec} A$ since we assumed the fibre $\operatorname{Spec} R^s_\Sigma \to \operatorname{Spec} \mathcal{O}$ at $(0,0,0)$ is unramified away from the closed point. We therefore let $U \neq \emptyset$ denote the (open) complement of the support of M, and $V = \beta^{-1}(U) \subset \operatorname{Spec} R_\Sigma$, claiming this gives the set we want in part (3) of the theorem.

Firstly, since $M = 0$ on U, $V \to U$ is unramified, and also finite flat by (1) so it is finite étale. Since $U \to \operatorname{Spec} \mathcal{O}$ is visibly formally smooth, we deduce that $V \to \operatorname{Spec} \mathcal{O}$ is formally smooth also. From the construction it is also clear that V contains all the points of $\operatorname{Spec} R^s_\Sigma[1/p]$. Since β is finite flat it sends open sets to open sets, so the density of U in (the connected) $\operatorname{Spec} A$ implies density of V in $\operatorname{Spec} R_\Sigma$, in particular V restricts to a dense open subset V_i in each component. Moreover, since each $V_i \to U$ is finite étale of some positive degree, the fibre above $(0,0,0)$ cannot be empty, so each component contains a point of $\operatorname{Spec} R^s_\Sigma[1/p]$.

From this, one can check reducedness by choosing for each component some f_i vanishing on the complement of V_i. Then $f = f_1 + ... + f_N$ is not a zero-divisor, and so $R_\Sigma \subset R_\Sigma[f^{-1}]$, with $\operatorname{Spec} R_\Sigma[f^{-1}]$ formally smooth over E and in particular reduced. Thus the main theorem of this section is proved.

Recall that we are trying to prove modular points are Zariski dense in $\operatorname{Spec} R_\Sigma$. The utility of our theorem is the following. Suppose the modular points are not dense, so there is some nonzero function $\tau \in R_\Sigma$ vanishing on all the modular points. We have shown (via our small $R = \mathbb{T}$ theorem) that each component has a modular point, so it is harmless

[4]See the appendix for a recap of some of this commutative algebra.

to assume $\operatorname{Spec} R_\Sigma$ is a domain on which τ is not identically zero. Passing to the rigid analytic generic fibre associated to $\operatorname{Spf} R_\Sigma$, the above theorem tells us that the modular points we have already found have neighbourhoods that look like 3-dimensional balls, and τ cuts out an algebraic hypersurface in such a ball on which all other modular points are constrained to lie. In the next section, we will think about modular forms and show that the p-adic analytic modular locus behaves too wildly to be constrained to such a simple subspace, and this contradiction proves the density result.

3. Coleman families, twins, and the infinite fern

In this section we follow [7] as well as [1, 5], to use Coleman's results on families of overconvergent modular forms to prove the following theorem.

Theorem 3.1. *Let R_Σ be a Galois deformation ring of level pN. Suppose we have a point $P \in \operatorname{Spec} R_\Sigma[1/p]$ given by a modular form f of some level pN' new away from p, and of noncritical slope. Suppose further that P has a rigid neighbourhood $B \subset \operatorname{Spf}(R_\Sigma)^{an}$ that is isomorphic to a 3-dimensional ball. Then any function τ on B vanishing on every modular point is identically zero.*

3.1. Analytic families of modular forms.

Let X denote the ball about P inside the rigid generic fibre of $\operatorname{Spf} R_\Sigma$. An analytic family of modular forms is given by two pieces of data.

- An analytic map $f : D \to X$ where $D \subset \mathbb{Z}_p$ is some rigid analytic disc.
- A further analytic nowhere vanishing function $u : D \to \mathbb{Z}_p$.

These data are said to give an analytic family of modular forms if there is a topologically dense arithmetic progression $\mathcal{K} \subset D \cap \mathbb{Z}$ such that for each $k \in \mathcal{K}$, $f(k)$ is the Galois representation coming from a modular form of weight k and level pN, and $u(k)$ is its U_p eigenvalue.

To convince oneself the existence of such things isn't utterly implausible, one should keep in mind the example of the Eisenstein series

$$E_{2k}(q) = -B_{2k}/2k + \sum_i \sigma_{2k-1}(i) q^i.$$

Since for any prime l, $k \mapsto 1 + l^k$ is a p-adically continuous function, one is able to show (using an argument going back to Serre) that this family can be p-adically interpolated in the variable k. I suppose one could translate this into a situation precisely in the setup above by considering this (for $k \equiv 2 \mod (p-1)$) as a locus in the deformation space of $\bar{\rho} = 1 \oplus \bar{\epsilon}$ and by taking some corresponding modular forms of level p.

The crucial result we will use due to Coleman says that such families are abundant in nature. Say that a cuspidal eigenform f of level pN and weight k_0 is *Coleman*[5] if:

- It has slope $\alpha < k_0 - 1$.
- The slope is not equal to $(k_0 - 1)/2$.

[5] This isn't standard terminology. The phrase "noncritical slope" is not far off.

- It is new away from p, in the sense that there does not exist $N'|N$ such that f comes from level pN'.

Theorem 3.2 (Coleman's existence of p-adic analytic families of modular forms)**.** *Suppose f is a Coleman modular form and gives rise to a point on X. Then there is a constant slope α p-adic analytic family of modular forms containing f.*

3.2. **Twins and the infinite fern.** We summarise some basic facts from [9] about modular forms that will allow us to go to town and make lots of Coleman families. Suppose f is an eigenform of level pN, of nebentypus ψ and U_p eigenvalue λ. By calculations of Ogg, one knows that unless $\lambda^2 = \psi(p)p^{k-2}$, f is not a newform but rather comes from a form ϕ of level N. In this situation, one knows that λ is a root of the polynomial $X^2 - a_pX + \epsilon(p)p^{k-1} = 0$ where a_p is the T_p eigenvalue of ϕ. However, the other root λ' of this polynomial also gives rise to an oldform f' of level pN with the same Galois representation as f but the different U_p-eigenvalue λ'. These two eigenforms span the space of such oldforms. We say that f and f' are *twins*.

Consider the following procedure. Start off with a Coleman modular form, and put it in a p-adic family. This family contains infinitely many points $f^{(k)}$ which are also Coleman modular forms[6], and where the slope is not equal to $(k-2)/2$. At each such point, we have a twin, and by taking the weight sufficiently large, the twin has slope $\alpha' = k - 1 - \alpha > \alpha$. We can now consider the point in deformation space coming from $f^{(k)}$ as actually coming from its twin, and then put a Coleman family through it of slope α'. The following lemma guarantees that this procedure spreads out the modular locus in a new direction.

Lemma 3.3. *Two Coleman families of different slopes $\alpha \neq \alpha'$ intersect in at most one point.*

Proof. An intersection point is given by a pair of modular forms f, f' (one coming from each family) whose Galois representations are equal. Since the slopes are different, we conclude that f and f' are twins, and so in particular $\alpha + \alpha' = k - 1$. This equation determines the weight k, and Coleman families are 1-1 over weight space so the lemma is proved. $\square$

This shows that at each of our infinitely many points, when we draw a new Coleman family it goes in a new direction. One can then iterate this procedure to obtain what Mazur calls the "infinite fern." One obtains in this way a really fun picture, but even better, we can use it to prove density of modular points in deformation space.

3.3. **Proof of the density theorem.** Starting with a Coleman point $x_0 = x(f_0)$ in X, we now give the argument proving density of the modular locus. We have some function τ vanishing on all the modular points, and will assume for contradiction that $\tau \neq 0$.

Firstly, consider the Hodge-Tate null space $X^0 \subset X$ consisting of points where one of the Hodge-Tate-Sen weights is zero. All points coming directly from eigenforms land in X^0

[6] I'm cheating a little bit: one needs to check these points are new away from p, and I'm not completely sure how to do this. Mazur's article [9] restricts attention to level 1.

(having Hodge-Tate weights $(0, k-1)$). We can also consider the space (isomorphic to a 1-dimensional ball) of wild characters Ψ and have an obvious "twist by a character" map

$$\Psi \times X^0 \to X.$$

This is surjective, and away from the locus where HT weights *both* vanish, is 2:1 and unramified.

Now, take a Coleman family $C \subset X^0$ through x_0, and shrinking if necessary we may assume it does not intersect the locus where both weights vanish, and that the restriction $\Psi \times C \to X$ of the above map is an immersion. Let $M \subset X$ be the two-dimensional submanifold that is its image. Now, by assumption $M \subset \{\tau = 0\}$. The ring of functions on a disc is a unique factorisation domain, so we can factor $\tau = \tau_1^{e_1} ... \tau_m^{e_m}$ into irreducibles in a unique manner. Shrinking M slightly and relabelling if necessary, we may further assume that in fact $M = M_1 := \{\tau_1 = 0\}$. We also may assume that C lies in every $M_j := \{\tau_j = 0\}$: if not, $C \cap M_j$ is finite and so after shrinking C is empty, whence it is harmless to replace τ by $\tau/\tau_j^{e_j}$. We will proceed by induction on m, so it will suffice to find a new family C' of modular points that doesn't lie inside M_1.

Firstly, let us note that $M \cap X_0 = C \cup C'$ where C' is the conjugate curve of C consisting of representations $\rho' = \rho \otimes (\det \rho)^{-1}_{wild}$ where $\rho \in C$. In particular it is a union of two curves. Now consider the needles $C^{(k)}$ of the fern manufactured from the Colman points of C. There are infinitely many of them, and they have different slopes from C, so intersect C in at most one point. Suppose for contradiction they all lie in M. Then they must all land in C'. But for two different $k \neq k'$ the curves $C^{(k)}$ and $C^{(k')}$ also have different slopes so again intersect in at most one point. Being open subsets of C' it is not possible to intersect in a single point, so we conclude they are all disjoint. Viewing this situation inside weight space, we have a dense set of points $S \subset \mathbb{Z}_p$ together with neighbourhoods U_s of s such that the U_s are pairwise disjoint. But this is clearly impossible. This contradiction gives the existence of a curve lying outside $M = M_1$, proving the inductive step and hence the theorem.

4. Appendix: Background topics

Here we include brief explanations of technical devices that are important in the proof and might be unfamiliar to some readers, but were too 'standard' or diversionary to include in the full text.

4.1. **Regular sequences and the Koszul complex.** Let R be a commutative ring. A *regular sequence* $a_1, a_2, ..., a_n$ in R is a sequence of elements such that $a_i \in A/(a_1, .., a_{i-1})$ is not a zero divisor for all i. For example, if R is a regular local Noetherian ring, any coordinate system for R gives a regular sequence (and in general for local Noetherian rings provided no a_i is a unit the notion of regularity doesn't depend on the ordering). Flat images of regular sequences are regular.

We say a closed immersion of locally Noetherian schemes $X \hookrightarrow Y$ is called a *regular immersion* if on each stalk the ideal defining the immersion is generated by a regular sequence. Say a morphism of locally Noetherian schemes $f : X \to Y$ is a local complete intersection if it is finite type and locally on X it can be factored as a regular immersion followed by a smooth map. This property is stable under composition and flat base change. It can also be checked on fibres $\operatorname{Spec} k(y) \to Y$.

The key facts are that any finite type map between regular locally Noetherian schemes is automatically lci, and that any closed immersion that is lci is automatically a regular immersion.

In studying the geometry of deformation rings we need the following lemma.

Lemma 4.1. *Let R be a complete noetherian local $\mathcal{O}$-algebra of dimension $m+1$. Then $f_1, ..., f_m, l$ are a regular sequence if and only if the map they induce*

$$\mathcal{O}[[x_1, ...x_m]] \to R$$

is finite flat.

If the given map is flat, then since flat images of regular sequences are regular, we are done. The converse is the more interesting statement.

We introduce the Koszul complex. Let R be a commutative ring, and $x \in R$ some element. The (homological) Koszul complex of (x, R) is just the two term complex

$$K(x, R)_\bullet := 0 \to R \overset{\times x}{\to} R \to 0.$$

If one now takes a collection $\{x_1, ..., x_n\}$ of elements, its Kozsul complex is just the the tensor product of all these

$$K(x_1, ..., x_n; R)_\bullet = K(x_1; R)_\bullet \otimes ... \otimes K(x_n; R)_\bullet$$

inside the category of bounded length chain complexes.

Recall that the differentials are defined in such a situation by $d(a \otimes b) = da \otimes b + a \otimes (-1)^{dega} db$ so one has to be careful about signs when computing with such objects. The key relationship to regular sequences is the following.

Proposition 4.2. *If $x_1, ..., x_n$ is regular, then the Kozsul complex $K(x_1, ..., x_n; R)_\bullet$ gives a free resolution of $R/(x_1, ..., x_n)$ (i.e. is exact outside degree zero).*

Let us use this to prove our lemma. Let $A = \mathcal{O}[[x_1, ...x_m]] \to R$ and we wish to show that $Tor_1^A(R, k) = 0$ (which immediately implies $A \to R$ is flat). Since $x_1, .., x_m, l$ is a regular sequence in A, $K(x_1, ..., x_n, p; A)_\bullet$ is a free resolution of k as an A-algebra, so we can use it to compute Tors. In particular, $K(x_1, ..., x_n, p; R)_\bullet = Tor_\bullet^A(R, k)$, and using the proposition again from the assumption that $f_1, ..., f_m, l$ is a regular sequence we conclude the higher Tors vanish.

We note that finiteness follows immediately from the dimension conditions imposed and Nakayama's lemma (regularity of the sequence implies $R/(f_1, ..., f_m, p)$ is Artinian, hence finite over k).

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