CYCLOTOMIC FIELDS PROBLEM SHEET 2

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(1) Factor 2 and 3 into prime ideals of $\mathbb{Z}[\sqrt{-5}]$. Check that 2 ramifies but 3 does not. Show that the class number of $\mathbb{Q}(\sqrt{-5})$ is divisible by 2.

(2) Following the method from lectures, show that the class number of $\mathbb{Q}(\sqrt{-2})$ is 1. Use this to solve

$$y^2 = x^3 - 2$$

for $x, y \in \mathbb{Z}$.

(3) Calculate the unit group of $\mathbb{Q}(\sqrt{11})$. You should give a formula for a general unit and prove there are no others.

(4) Let p, q, r be distinct primes. Show that $\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$ is a degree 8 extension of $\mathbb{Q}$.

(5) Using the arithmetic and Galois theory of $\mathbb{Q}(\zeta_8)/\mathbb{Q}$, derive a formula for the primes p with the property that 2 is a square modulo p.

(6) Show that any quadratic field $\mathbb{Q}(\sqrt{d})$ is contained in some cyclotomic field (without using Kronecker-Weber!).

(7) (*) Show that the class number of $\mathbb{Q}(\zeta_{23})$ is divisible by 3.