CYCLOTOMIC FIELDS PROBLEM SHEET 4

TOM LOVERING (LOVERING@FAS.HARVARD.EDU)

(1) Let X be the group of all Dirichlet characters of conductor dividing p, $a \in \mathbb{Z}$. Show that
$$\sum_{\chi \in X} \chi(a) = \begin{cases} p & \text{if } a \equiv 1 \mod p \\ 1 & \text{if } a \equiv 0 \mod p \\ 0 & \text{otherwise.} \end{cases}$$

(2) Consider χ a Dirichlet character of conductor $f > 1$. Show that
$$\sum_{a=1}^{f} \chi(a) = 0.$$

(3) Suppose two Dirichlet characters χ, ψ have conductors f_χ, f_ψ which are coprime. Show that the conductor of $\chi.\psi$ is $f_\chi f_\psi$.

(4) Take your favourite Dirichlet character χ of conductor at least 5, and compute $B_{1,\chi}, B_{2,\chi}$ and $B_{3,\chi}$.

(5) Compute the regulator R_K for $K = \mathbb{Q}(\sqrt{11})$.

(6) Find an upper bound for the regulator R_K for $K = \mathbb{Q}(\zeta_5)$.

(7) (*) Let p be an odd prime, m even with $2 \leq m \leq p-3$. For any integer a let $r(a) \in \{0, ..., p-1\}$ denote its least nonnegative residue mod p. Let g be an integer which is a primitive root mod p but satisfies $g^{p-1} \equiv 1 \mod p^2$ (i.e. has exact order $p-1$ both mod p and mod p^2). Prove that
$$1^m + ... + (p-1)^m \equiv m \sum_{a=1}^{p-1} g^{(m-1)a} r(g^a) \mod p^2.$$