CYCLOTOMIC FIELDS PROBLEM SHEET 5

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(1) Compute the group of Dirichlet characters associated to (i.e. factoring through the Galois group of) the field $\mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$. What is the discriminant of this field?

(2) Carefully compute (after suitably dividing into cases) $\mu_K = (\mathcal{O}_K^*)_{tors}$ for $K = \mathbb{Q}(\zeta_n)$ and $K^+ = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

(3) Show that if $\alpha \in \mathcal{O}_K$ is such that for all $\tau : K \hookrightarrow \mathbb{C}$, we have that $|\tau(\alpha)| = 1$, then the coefficients of the minimal polynomial for α are bounded in terms of the degree $[K : \mathbb{Q}]$. Deduce that for any such α there is $n \in \mathbb{N}$ such that $\alpha^n = 1$.

(4) Prove the conductor-discriminant formula in the complex case. You will need to use (can you also prove?) the identity

$$\Gamma(s/2)\Gamma((s+1)/2) = 2^{1-s}\sqrt{\pi}\Gamma(s).$$

(5) (Class numbers of quadratic fields) For $m \in \mathbb{Z}$, let $K = \mathbb{Q}(\sqrt{m})$.
- Recall or compute ω_K, R_K and D_K.
- Identify the unique nonprincipal ($\chi \neq 1$) Dirichlet character χ associated to K (hint: the conductor-discriminant formula tells you the smallest cyclotomic field you need to embed K into).
- Derive a formula for the class number h_K in terms of $L(1, \chi)$.
- In the case where $m < 0$, use generalised Bernoulli numbers to get a closed formula for h_K, and use it to compute h_K for your favourite example.